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ILL–POSEDNESS FOR THE CAUCHY PROBLEM FOR A FAMILY OF TWO–DIMENSIONAL BENJAMIN–ONO EQUATION

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ABSTRACT

In this paper going to be showing for all $s \in \mathbb{R}$, when $1 \leq \alpha < 2$, the ill-posedness for the Cauchy problem 1.

$$\begin{cases} u_t + D_x^{\alpha} u_x + \mathcal{H}_x u_{yy} + u u_x = 0, & (x, y) \in \mathbb{R}^2, \quad t \geq 0, \\ u(x, y, 0) = u_0(x, y), \end{cases} \tag{1}$$

where $\mathcal{H}_x$ is the Hilbert transform in the first variable, and $1 \leq \alpha \leq 2$.

***KEYWORDS:** Phrases & Math Subject Classification: Ill-posedness, Bidimensional, Benjamin-Ono Equation, Picard Scheme, 35A01, 35A02, 35A22, 35A23 & 35B30*

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Original Article

1. INTRODUCTION

Is known that The Picard iterative scheme cannot be used to prove the local well-posedness (LWP) for the initial value problem (IVP) with the Benjamin-Ono equation 2.

$$\begin{cases} u_t + \mathcal{H} u_{xx} + u u_x = 0, & x, t \in \mathbb{R} \\ u(0, x) = u_0. \end{cases} \tag{2}$$

Also, when the equation has a general nonlinear part, in [8] using the Picard scheme the authors proved the LWP for the IVP 3

$$\begin{cases} u_t + \mathcal{H} u_{xx} + u^k u_x = 0, & x, t \in \mathbb{R} \\ u(0, x) = u_0. \end{cases} \tag{3}$$

Using the ideas exposed in [8] Kenig-Martel and Robiano in [7] proved the LWP for

$$\begin{cases} u_t + D_x^{\alpha} u_x + u^k u_x = 0, & x, t \in \mathbb{R} \\ u(0, x) = u_0. & \text{with} 1 \leq \alpha \leq 2 \end{cases} \tag{4}$$

It is very interesting that cannot be used the same argument for the bidimensional extension 1, because in this paper gonna be show a bounded sequence of functions in H^{∞} over cannot be applied the Picard Scheme.

In [2] shows LWP using energy estimates for 1 when $0 \leq \alpha \leq 1$ and prove the ill–posedness for all $s \in \mathbb{R}$. For LWP of 1 the reader can see [3] and [11]

2. ILLPOSEDNESS

Using the ideas in [10], and [2] here prove that the IVP 1 cannot be solved by a Picard iterative scheme based on the Duhamel's formula. In other words, if

$$u(t) = U_\alpha(t)\phi - \int_0^t U_\alpha(t-t')[u(t')u_x(t')]dt' \tag{5}$$

That is the integral equation corresponding to the IVP 1 with initial datum ϕ, in wich

$$(U_\alpha(t)\phi)^\wedge(\xi,\eta) := e^{-it\left(|\xi|^\alpha \xi + sgn(\xi)\eta^2\right)}\hat{\phi}(\xi,\eta), \tag{6}$$

the following assertion holds.

Theorem 1 *Let* $\alpha \in [1,2)$, $s \in \mathbb{R}$ *and* $T > 0$. *Then, there does not exist a space* X_T *continuously embedded in* $C([0,T]; H^s(\mathbb{R}^2))$ *such that there exist* $C > 0$ *with*

$$\left||U_\alpha(t)\phi|\right|_{X_T} \leq C\left||\phi|\right|_{H^s(\mathbb{R}^2)} \tag{7}$$

and

$$\left|\left|\int_0^t U_\alpha(t-t')(u(t')u_x(t')\,dt')\right|\right|_{X_T} \leq C\left||u|\right|_{X_T}^2, \quad u \in X_T \tag{8}$$

Let us observe that the conditions 7 and 8 are necessary to apply the Picard iterative method scheme in the integral equation 5

Proof. Reasoning as in [10] can be supposed there exist a space X_T such that 7 and 8 holds. Let us define in 8 $u := U_\alpha(t)\phi$. Then

$$\left|\left|\int_0^t U_\alpha(t-t')[(U_\alpha(t')\phi)(U_\alpha(t')\phi_x)]\,dt'\right|\right|_{X_T} \leq C\left||U_\alpha(t)\phi|\right|_{X_T}^2$$

In this way, using 7 and the fact that $X_T{}^\circ C([0,T]; H^s(\mathbb{R}^2))$, can be concluded for each $t \in [0,T]$

$$\left|\left|\int_0^t U_\alpha(t-t')[(U_\alpha(t')\phi)(U_\alpha(t')\phi_x)]\,dt'\right|\right|_{H^s(\mathbb{R}^2)} \lesssim \left||U_\alpha(t)\phi|\right|_{H^s(\mathbb{R}^2)}^2 \tag{9}$$

Will show 9 is not true, exhibiting a bounded sequence $\{\phi_N\}_{N\in\mathbb{N}} \in H^s(\mathbb{R}^2)$, such that for $t \in [T/2, T]$,

$$\lim_{N\to\infty} \left|\left|\int_0^t U_\alpha(t-t')[(U_\alpha(t')\phi_N)(U_\alpha(t')(\phi_N)_x)]\,dt'\right|\right|_{H^s(\mathbb{R}^2)} = \infty$$

Let us take $\phi \equiv \phi_N$ defined through the Fourier transform as follows

$$\begin{aligned}&\hat{\phi}(\xi,\eta) = \beta^{-\frac{1+\delta}{2}}\chi_{I_1}(\xi,\eta) + \beta^{-\frac{1+\delta}{2}}N^{-s}\chi_{I_2}(\xi,\eta),\\ &N \gg 1, \quad 0 < \beta \ll 1, \quad \delta > 0, \quad \delta = \delta(\alpha)\end{aligned}$$

Where $I_1 = [\beta/2, \beta] \times [0, \beta^\delta]$ and $I_2 = [N, N+\beta] \times [0, \beta^\delta]$, β to be precise later, and χ_A denotes the characteristic function of the set A. Let us note that

$$\begin{aligned}\left||\phi|\right|_{H^s(\mathbb{R}^2)}^2 &= \int_{\mathbb{R}^2} (1+\xi^2+\eta^2)^s\left|\hat{\phi}(\xi,\eta)\right|^2 d\xi\eta\\ &= \int_{I_1} (1+\xi^2+\eta^2)^s\beta^{-(1+\delta)}\,d\xi\eta +\\ &+ \int_{I_2} (1+\xi^2+\eta^2)^s\beta^{-(1+\delta)}N^{-2s}\,d\xi\eta\sim\\ &\sim \int_{\beta/2}^{\beta}\int_0^{\beta^\delta} \beta^{-(1+\delta)}\,d\xi\eta + \int_N^{N+\beta}\int_0^{\beta^\delta} N^{2s}\beta^{-(1+\delta)}N^{-2s}\,d\xi\eta\sim C\end{aligned} \tag{10}$$

Moreover, is defined $\rho(\xi,\eta)=-(|\xi|^{\alpha}\xi+sgn(\xi)\eta^2)$, $\theta(\xi,\eta,\xi_1,\eta_1)=\rho(\xi_1,\eta_1)+\rho(\xi-\xi_1,\eta-\eta_1)-\rho(\xi,\eta)$, and $\psi(\xi,\eta,\xi_1,\eta_1)=\hat{\phi}(\xi,\eta)\hat{\phi}(\xi-\xi_1,\eta-\eta_1)$. Then it can be easily seen that

$$\int_0^t U_\alpha(t-t')[(U_\alpha(t')\phi)(U_\alpha(t')\phi_x)]\,dt'=$$
$$=C\int_{\mathbb{R}^2} e^{i(x\xi+y\eta)}e^{it\rho(\xi,\eta)}\xi\left[\int_{\mathbb{R}2}\psi(\xi,\eta,\xi_1,\eta_2)\frac{e^{it\theta(\xi,\eta,\xi_1,\eta_1)}-1}{\theta(\xi,\eta,\xi_1,\eta_1)}\,d\xi_1\eta_1\right]d\xi\eta$$

Considering the following four sets

$$I_{ij}(\xi,\eta)=\{(\xi_1,\eta_1)\in\mathbb{R}^2 \mid (\xi_1,\eta_1)\in I_i,\ (\xi-\xi_1,\eta-\eta_1)\in I_j\},\quad i,j\in\{1,2\} \tag{11}$$

To obtain $\psi(\xi,\eta,\xi_1,\eta_1)\neq 0$, it is required $(\xi_1,\eta_1)\in I_{ij}(\xi,\eta)$ for someone of this four sets. Using the notation $\theta\equiv\theta(\xi,\eta,\xi_1,\eta_1)$, holds that

$$\begin{aligned}\int_0^t U_\alpha(t-t')\ &[(U_\alpha(t')\phi)(U_\alpha(t')\phi_x)]\,dt'=\\ &=C\int_{\mathbb{R}^2} e^{i(x\xi+y\eta)}e^{it\rho(\xi,\eta)}\xi\left[\int_{A_{11}(\xi,\eta)}\beta^{-(1+\delta)}\frac{e^{it\theta}-1}{\theta}\,d\xi_1\eta_1+\right.\\ &+\int_{A_{22}(\xi,\eta)}\beta^{-(1+\delta)}N^{-2s}\frac{e^{it\theta}-1}{\theta}\,d\xi_1\eta_1+\\ &\left.+\int_{A_{12}\cup A_{21}(\xi,\eta)}\beta^{-(1+\delta)}N^{-s}\frac{e^{it\theta}-1}{\theta}\,d\xi_1\eta_1\right]d\xi\eta\end{aligned}$$

Define

$$\begin{aligned}f_1(x,y,t)&:=\int_{\mathbb{R}^2}\frac{C\xi e^{i(x\xi+y\eta)}e^{it\rho(\xi,\eta)}}{\beta^{(1+\delta)}}\left[\int_{A_{11}(\xi,\eta)}\frac{e^{it\theta}-1}{\theta}\,d\xi_1\eta_1\right]d\xi\eta\\ f_2(x,y,t)&:=\int_{\mathbb{R}^2}\frac{C\xi e^{i(x\xi+y\eta)}e^{it\rho(\xi,\eta)}}{\beta^{(1+\delta)}N^{2s}}\left[\int_{A_{22}(\xi,\eta)}\frac{e^{it\theta}-1}{\theta}\,d\xi_1\eta_1\right]d\xi\eta\\ f_3(x,y,t)&:=\int_{\mathbb{R}^2}\frac{C\xi e^{i(x\xi+y\eta)}e^{it\rho(\xi,\eta)}}{\beta^{(1+\delta)}N^{s}}\left[\int_{A_{12}\cup A_{21}(\xi,\eta)}\frac{e^{it\theta}-1}{\theta}\,d\xi_1\eta_1\right]d\xi\eta\end{aligned} \tag{12}$$

$$\left\{\int_0^t U_\alpha(t-t')[(U_\alpha(t')\phi)(U_\alpha(t')\phi_x)]\,dt'\right\}(x,y)=$$
$$=f_1(x,y,t)+f_2(x,y,t)+f_3(x,y,t)$$

and will can see that

$$\begin{aligned}supp(\widehat{f_1})&\subset[\beta,2\beta]\times[0,2\beta^{\delta}]\\ supp(\widehat{f_2})&\subset[2N,2N+2\beta]\times[0,2\beta^{\delta}]\\ supp(\widehat{f_3})&\subset\left[N+\frac{\beta}{2},N+2\beta\right]\times[0,2\beta^{\delta}]\end{aligned}$$

The supports are mutually disjoint. We have that

$$\left\|\int_0^t U_\alpha(t-t')[(U_\alpha(t')\phi)(U_\alpha(t')\phi_x)]\,dt'\right\|_{H^s(\mathbb{R}^2)}\geq\|f_i(\cdot,\cdot,t)\|_{H^s(\mathbb{R}^2)} \tag{13}$$

Making the change of variables $\xi_2:=\xi-\xi_1$ and $\eta_2:=\eta-\eta_1$ and taking into account that $\theta(\xi,\eta,\xi_2,\eta_2)=\theta(\xi,\eta,\xi_2,\eta_2)$, it is easy to see that

$$\int_{A_{21}(\xi,\eta)}\frac{e^{it\theta(\xi,\eta,\xi_1,\eta_1)}-1}{\theta(\xi,\eta,\xi_1,\eta_1)}\,d\xi_1\eta_1=\int_{A_{12}(\xi,\eta)}\frac{e^{it\theta(\xi,\eta,\xi_2,\eta_2)}-1}{\theta(\xi,\eta,\xi_2,\eta_2)}\,d\xi_2\eta_2$$

Therefore,

$$
\begin{aligned}
&f_3(x,y,t) = \\
&= \frac{2C}{\beta^{(1+\delta)}N^s}\int_{\mathbb{R}^2} \xi e^{i(x\xi+y\eta)} e^{it\rho(\xi,\eta)}\left[\int_{A_{12}(\xi,\eta)} \frac{e^{it\theta}-1}{\theta}\, d\xi_1\eta_1\right] d\xi\eta
\end{aligned}
$$

and,

$$
\begin{aligned}
&||f_3(\cdot,\cdot,t)||^2_{H^s(\mathbb{R}^2)} = \\
&= \int_{supp(\widehat{f_3})} (1+\xi^2+\eta^2)^s \frac{C\xi^2}{\beta^{2(1+\delta)}N^{2s}}\left|\int_{A_{12}(\xi,\eta)} \frac{e^{it\theta}-1}{\theta}\, d\xi_1\eta_1\right|^2 d\xi\eta
\end{aligned}
$$

taking account the support of $\widehat{f_3}$, i.e. $(\xi,\eta) \in \left[N+\frac{\beta}{2}, N+2\beta\right] \times \left[0,2\beta^\delta\right]$

$$
||f_3(\cdot,\cdot,t)||^2_{H^s(\mathbb{R}^2)} \sim \frac{CN^{2s}N^2}{\beta^{2(1+\delta)}N^{2s}}\int_{supp(\widehat{f_3})}\left|\int_{A_{12}(\xi,\eta)} \frac{e^{it\theta}-1}{\theta}\, d\xi_1\eta_1\right|^2 d\xi\eta
$$

take β such a way that $\beta N^\alpha = N^{-\varepsilon}$, with $0<\varepsilon \ll 1$, it follows that

$$
\begin{aligned}
\theta(\xi,\eta,\xi_1,\eta_1) &= \rho(\xi_1,\eta_1) + \rho(\xi-\xi_1,\eta-\eta_1) - \rho(\xi,\eta) \\
&= -\xi_1^{\alpha+1} - (\xi-\xi_1)^{\alpha+1} + \xi^{\alpha+1} + 2\eta_1(\eta-\eta_1) \\
&= -\xi_1^{\alpha+1} - [\xi^{\alpha+1} - (\alpha+1)\xi^\alpha\xi_1 + \\
&\quad +o(\xi_1)] + \xi^{\alpha+1} + 2\eta_1(\eta-\eta_1) \\
&\qquad \left(\text{where} \lim_{\xi_1\to 0}\frac{o(\xi_1)}{\xi_1} = 0\right) \\
&= (\alpha+1)\xi^\alpha\xi_1 - \xi_1^{\alpha+1} - o(\xi_1) + 2\eta_1(\eta-\eta_1) \\
&= \xi_1\left[(\alpha+1)\xi^\alpha - \xi_1^\alpha - \frac{o(\xi_1)}{\xi_1}\right] + 2\eta_1(\eta-\eta_1) \\
&\sim \beta N^\alpha,
\end{aligned}
$$

Taking into account that $\frac{1-\cos(\gamma)}{\gamma} \sim \gamma$ for $0<\gamma \ll 1$, then if $[T/2,T]$, follows:

$$
\begin{aligned}
||f_3(\cdot,\cdot,t)||_{H^s(\mathbb{R}^2)} &\gtrsim Ct^2\frac{N^2}{\beta^{2(1+\delta)}}\int_{supp(\widehat{f_3})}\left|Re\int_{A_{12}(\xi,\eta)} \frac{e^{it\theta}-1}{t\theta}\, d\xi_1\eta_1\right|^2 d\xi\eta \\
&\gtrsim \frac{Ct^2N^2}{\beta^{2(1+\delta)}}\int_{supp(\widehat{f_3})}\left|\int_{A_{12}(\xi,\eta)} \frac{1-\cos(t\theta)}{t\theta}\, d\xi_1\eta_1\right|^2 d\xi\eta \\
&\gtrsim \frac{CT^2N^2}{\beta^{2(1+\delta)}}\int_{supp(\widehat{f_3})}\beta^2N^{2\alpha}\left|\int_{A_{12}(\xi,\eta)} d\xi_1\eta_1\right|^2 d\xi\eta. \\
&\gtrsim \frac{CT^2N^2}{\beta^{2(1+\delta)}}\beta^2N^{2\alpha}\left(\beta^{1+\delta}\right)^3 = N^{2+2\alpha}\beta^{2+(1+\delta)} = \\
&= N^{2+2\alpha}(N^{-\alpha-\varepsilon})^{(3+\delta)} \\
&= N^{2-(1+\delta)\alpha-(3+\delta)\varepsilon} \to \infty, \quad \text{if } \alpha < \frac{2}{1+\delta}
\end{aligned}
$$

As we can see, if $\delta \to 0$, then α would be near to 2

Theorem 2 *Fix* $s \in \mathbb{R}$. *Then there does not exist a* $T>0$, *such that 1 admits a unique local solution defined on* $[0,T]$ *and such that the flow map data–solution*

$$\phi \to u(t), \quad t \in [0,T]$$

for 1 is C^2 differentiable at zero from $H^s(\mathbb{R}^2)$ to $H^s(\mathbb{R}^2)$

Proof. Consider the Cauchy problem

$$\begin{cases} u_t + D^\alpha u_x + \mathcal{H} u_{yy} + u u_x = 0, \\ u(x,y,0) = \gamma \phi(x,y), \qquad \gamma \ll 1, \quad \phi \in H^s(\mathbb{R}^2) \end{cases} \tag{14}$$

Suppose that $u_\gamma(x,y,t)$ is a local solution of 14 and that the flow map is C^2 at the origin from $H^s(\mathbb{R}^2)$ to $H^s(\mathbb{R}^2)$. Then

$$\left.\frac{\partial u_\gamma}{\partial \gamma^2}\right|_{\gamma=0} = -2\int_0^t U_\alpha(t-t')[(U_\alpha(t')\phi)(U_\alpha(t')\phi_x)]\,dt'$$

The assumption of C^2 regularity yields

$$\left\|\int_0^t U_\alpha(t-t')[(U_\alpha(t')\phi)(U_\alpha(t')\phi_x)]\,dt'\right\|_{H^s(\mathbb{R}^2)} \lesssim \|U_\alpha(t)\phi\|^2_{H^s(\mathbb{R}^2)} \tag{15}$$

But the above estimate is 9, which has been shown to fail.

3. CONCLUSIONS

- Can't be posible to use the Picard Scheme to prove the local well posedness for the initial value problem (1)
- The map datum-solution is not at the origin, this result is known as ill-posedness

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